

A scalar-Einstein Wave.

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Abstract

Solution to the scalar-Einstein equations are found which contain both a scalar field and a gravitational wave; the Bell-Robinson tensor is used to give an indication of the solutions properties.

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1 Introduction

Spacetimes which contain both gravitational wave and have a scalar field obeying

$$Rs_{ab} \equiv R_{ab} - 2\phi_a\phi_b = 0, \quad (1)$$

are sought. There are at least *two* methods of attempting this. The *first* method is to choose a plane wave line element and investigate when its stress is that of a scalar field. In this case the space dependent Ricci components vanish, the Bell-Robinson tensor has only *u* components and the size of the Bell-Robinson tensor vanishes indicating null radiation. The *second* method

is to adjust the geometry. In this case the space dependent Ricci components are non-vanishing and all invariants including the size of the Bel-Robinson tensors can be expressed in terms of the Ricci scalar. The size of the Bel-Robinson tensor suggests spacelike energy propagation: whether this is good or bad or says more about the Bel-Robinson criterion or the spacetime itself are open questions. Calculations were done using `grtensor2/maple9` [2].

2 The plane wave.

The plane wave has line element [1]

$$ds^2 = W(u, y, z)du^2 + 2dudv + dy^2 + dz^2, \quad (2)$$

the determinant of the metric is $g = -1$ and the Kretschmann curvature invariant vanishes $K = RiemSq = 0$, the non-vanishing components of the Riemann tensor are given by

$$R_{uiuj} = -\frac{1}{2}W_{,ij} \quad (3)$$

where $i, j \dots = 1, 2, 3$. The non-vanishing component of the Ricci tensor is

$$R_{uu} = -\frac{1}{2}W_{,yy} - \frac{1}{2}W_{,zz}. \quad (4)$$

The Bel-Robinson tensor is an indicator of the amount of energy squared present and is defined by

$$Br_{cdef} \equiv C_{acdb}C_{.ef.}^{a. b} + *C_{acdb} * C_{.ef.}^{a. b}, \quad (5)$$

for the line element 2 it is found to be

$$Br_{uuuu} = \frac{1}{4}(W_{,zz} - W_{,yy}) + W_{,yz}^2, \quad (6)$$

the size of the Bel-Robinson tensor is

$$BS \equiv Br_{abcd}Br^{abcd} \quad (7)$$

and for the line element 2 it vanishes indicating that the direction of energy propagation is null. For a vacuum Ricci flat spacetime a choice of W is

$$W = (y^2 - z^2)f(u) - 2yzg(u) \quad (8)$$

where f, g are arbitrary twice differentiable functions of u . The Bach tensor is

$$B_{ab} \equiv 2C_{a..b}^{cd}R_{cd} + 4C_{a..b;cd}^{cd}, \quad (9)$$

and this tensor can be used in the expression for quadrtaic field equations

$$R_{ab} + bB_{ab} = 0. \quad (10)$$

For the line element 2 the Bach tensor has non-vanishing component

$$B_{uu} = W_{,yyyy} + 2W_{,yyzz} + W_{,zzzz}. \quad (11)$$

A solution to the field equations 10 is

$$W = \sin\left(\frac{y}{\sqrt{b}}\right) f_1(u) + \cos\left(\frac{y}{\sqrt{b}}\right) f_2(u) + \sin\left(\frac{z}{\sqrt{b}}\right) g_1(u) + \cos\left(\frac{z}{\sqrt{b}}\right) g_2(u). \quad (12)$$

For a scalar-Einstien solution one can choose

$$W = (ay^2 - bz^2)f(u) - 2cyzg(u), \quad (13)$$

giving

$$R_{uu} = (b - a)f = 2\phi_u^2, \quad (14)$$

however ϕ can have no y, z dependence as this would entail non-vanishing R_{yy} and in this sense the scalar field is nnot co-moving with the gravitational field.

3 Scalar-Einstein wave.

For a y, z dependent scalar-Einstein wave consider the line element

$$ds^2 = W(u, x, y)du^2 + 2A_3xydudv + A_1dx^2 + A_2dy^2, \quad \phi = \frac{1}{2} \ln\left(\frac{kx}{y}\right), \quad (15)$$

in the case of vanishing gravitational wave it is related to [3]. After subtracting off the scalar field ther remains the Ricci tensor component

$$Rs_{uu} = -\frac{1}{2A_2y^2} (W - yW_{,y} + y^2W_{,yy}) - \frac{1}{2A_1x^2} (W - xW_{,x} + x^2W_{,xx}). \quad (16)$$

The line element 15 is a scalar-Einstein solution when

$$\begin{aligned}
W &= \left(B_1 x \text{BesselJ} \left(0, \frac{x}{\sqrt{A_2}} \right) + B_2 x \text{BesselY} \left(0, \frac{x}{\sqrt{A_2}} \right) \right) \\
&\times \left(C_1 y \text{BesselJ} \left(0, \frac{y}{\sqrt{-A_1}} \right) + C_2 y \text{BesselY} \left(0, \frac{y}{\sqrt{-A_1}} \right) \right) f(u),
\end{aligned} \tag{17}$$

lowest order expansion suggests that the C_1 term is real but the C_2 term might be complex. A more simple solution to 16 is

$$H = (B_1 x + B_2 x \ln(x)) (C_1 y + C_2 y \ln(y)) f(u) \tag{18}$$

the invariants can be expressed in terms of the Ricci scalar

$$R = \frac{A_1 x^2 + A_2 y^2}{2A_1 A_2 x^2 y^2} \tag{19}$$

and are

$$\begin{aligned}
K &= 3R^2, \quad WeylSq = \frac{4}{3}R^2, \quad RicciSq = R^2, \quad BS = \frac{4}{9}R^4 \\
R_1 &= \frac{3}{16}R^2, \quad R_2 = \frac{3}{64}R^3, \quad R_3 = \frac{21}{1024}R^4, \quad W_{1R} = \frac{1}{6}R^2, \quad W_{2R} = \frac{1}{36}R^3, \\
M_{2R} &= M_3 = \frac{1}{96}R^4, \quad M_4 = \frac{1}{768}R^5, \quad M_{5R} = \frac{1}{576}R^5.
\end{aligned} \tag{20}$$

For this geometry the Bell-Robinson tensor 5 is complicated, however the size of it as given in 20 is simple and positive, suggesting spacelike propagation.

References

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- [3] Mark D. Roberts
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